A Beam Dynamics View on a Generalized Formulation of Spin Dynamics, Based on Topological Algebra, with Examples

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Here I rephrase some of the results of work[1–6] performed in several collaborations with **K.Heinemann & J.A.Ellison**[*], **D.P.Barber**[†], and **A.Kling**[‡] on a generalized look on spin dynamics and beam polarization in storage rings. It is done in a way that emphasizes the applicability of the concepts to real world polarized beams rather than presenting the results in their most general form. The latter view can be found in several articles on the ArXiv and will be published in refereed journals soon. I will introduce several "spin-related" systems, state some selected main results of the above mentioned work and then recover and compare some basic (and some not so basic) findings for the various systems in the light of our generalized approach.

1. Introduction

Here we will review some basics of spin/polarization dynamics in storage rings and define the dynamical systems that act as base model for this study.

We will employ a completely time-discrete picture, i.e. in the language of storage rings: we will fix an azimuth (location) in the ring and work with one-turn-maps only. We will fully neglect Stern-Gerlach type of back-reaction of the spins on the orbital trajectories, in other words the spin dynamics acts as passenger/spectator on top of the completely self-contained orbital motion.

1.1. *Spin/Orbit Dynamics*

We assume the orbital motion is in action angle variables J = const and ϕ advances linearly and uniformly. We generalize the model to d degrees of freedom ($2d$-dimensional phase space). Typically spin-orbit models have $d_0 = 1$ to 3. It is, however, often useful to increase d by the number of *independent* RF-elements (RF-dipoles, RF-solenoids, RF-quadrupoles, etc.) that act on spin inside the storage ring. We put the d phases on the d-torus $\mathcal{T}^d$ where we assume all phases in normalized units $\phi = \mu/(2\pi)$ so that the 1-torus can be viewed as $\mathcal{T} = \mathbb{R} \mod_1$. The orbital map M_ω advances the phases by the tune $\omega = \text{const} \in \mathcal{T}^d$.

$$M_\omega : \mathcal{T}^d \to \mathcal{T}^d, \phi \mapsto M_\omega(\phi) = \phi + \omega \mod_1 . \tag{1}$$

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In principle, more general (bijective) maps on the d-torus are permissible, but we consider this to be an unnecessary complication within the scope of this paper. We remind the reader that ω (and M_ω) are called *resonant* if a $k \equiv (k_0, \vec{k}) \in \mathbb{N}^{1+d}$ with $k \neq 0$ exists so that $k \cdot (1, \omega) = 0$. If M_ω is non-resonant, then for every $\phi_0 \in \mathcal{T}^d$ the sequence $\{M_\omega^n(\phi_0) : n \in \mathbb{Z}\}$ is dense in $\mathcal{T}^d$.

We assume the BMT-evolution for vector polarizations $\vec{s} \in \mathbb{R}^3$ and tensor Polarizations $\underline{\bar{s}} \in \mathbb{R}^{3\times 3}$ with $\underline{\bar{s}} = \underline{\bar{s}}^{\mathrm{T}}$ and $\mathrm{trace}\,\underline{\bar{s}} = 0$. That means that the one-turn-polarization transport along an orbital trajectory starting at $\phi \in \mathcal{T}^d$ is given by

$$\vec{s} \mapsto \underline{R}(\phi)\,\vec{s} \quad \& \quad \underline{\bar{s}} \mapsto \underline{R}(\phi)\,\underline{\bar{s}}\,\underline{R}(\phi)^{\mathrm{T}} \;, \tag{2}$$

where the function $\underline{R}$ is defined

$$\underline{R} : \mathcal{T}^d \to \mathbf{SO(3)} \;,\; \phi \mapsto \underline{R}(\phi) \;. \tag{3}$$

The complete one turn map of the polarization-orbit trajectory is

$$K \;:\; \begin{pmatrix} \phi \\ \vec{s}\;\text{(or)}\;\underline{\bar{s}} \end{pmatrix} \mapsto \begin{pmatrix} M_\omega(\phi) \\ \underline{R}(\phi)\,\vec{s}\;\text{(or)}\;\underline{R}(\phi)\,\underline{\bar{s}}\,\underline{R}(\phi)^{\mathrm{T}} \end{pmatrix} \;. \tag{4}$$

We define a "spin" ($\hat{s}$, $\underline{\check{s}}$) to be a "polarization" ($\vec{s}$, $\underline{\bar{s}}$) with norm 1 for a suitable norm which is invariant under $\mathbf{SO(3)}$ transport:

$$\|\hat{s}\|_2 := \sqrt{\hat{s}^{\mathrm{T}}\hat{s}} = 1 \quad \to \quad \|\underline{R}\,\vec{s}\|_2 = \|\vec{s}\|_2 \tag{5}$$

$$\|\underline{\check{s}}\|_{\mathrm{t}} := \sqrt{\mathrm{trace}\,(\underline{\check{s}}^{\mathrm{T}}\underline{\check{s}})} = 1 \quad \to \quad \|\underline{R}\,\underline{\bar{s}}\,\underline{R}^{\mathrm{T}}\|_{\mathrm{t}} = \|\underline{\bar{s}}\|_{\mathrm{t}} \;. \tag{6}$$

We therefore choose the one turn map of the spin-orbit trajectory identical to Eq. (4).

1.2. *Dynamics of Fields*

The time discrete dynamics of fields looks a little bit un-intuitive at first sight. So let's sidetrack to *time-continuous* spin dynamics for a tiny moment. In continuous time a vector polarization- or spin field is a field $\mathcal{T}^{d+1} \to \mathbb{R}^3$, $(\phi, \theta) \mapsto s(\phi, \theta)$, where θ is the generalized azimuth along the ring, so that, given $s(\cdot, \theta_0)$ at some θ_0, evaluating the field s evaluated at some $\theta_1 > \theta_0$ for some ϕ is equivalent to transporting s from θ_0 to θ_1 with the BMT-flow and starting from the backwards tracked ϕ. The BMT-flow in a storage ring is non-autonomous, but periodic so that the one turn map at each given reference azimuth is autonomous. Correspondingly, we define a *time-discrete* vector/tensor polarization/spin field (e.g. via the autonomous one turn map of a storage ring) as a *sequence* of fields $F_n := \vec{S}_n / \hat{S}_n$ / $\underline{F}_n := \underline{\bar{S}}_n / \underline{\check{S}}_n$, $n \in \mathbb{Z}$, on the torus: $F_n : \mathcal{T}^d \to \mathbb{R}^2$ / $\underline{F}_n : \mathcal{T}^d \to \mathbb{R}^{3x3}$ so that

$$\begin{aligned} F_{n+1}(M_\omega(\phi)) &= \underline{R}(\phi)\,F_n(\phi) \quad \Leftrightarrow \\ F_{n+1}(\phi) &= \underline{R}(M_\omega^{-1}(\phi))\,F_n(M_\omega^{-1}(\phi)) \end{aligned} \tag{7}$$

$$\begin{aligned} \underline{F}_{n+1}(M_\omega(\phi)) &= \underline{R}(\phi)\,\underline{F}_n(\phi)\,\underline{R}(\phi)^{\mathrm{T}} \quad \Leftrightarrow \\ \underline{F}_{n+1}(\phi) &= \underline{R}(M_\omega^{-1}(\phi))\,\underline{F}_n(M_\omega^{-1}(\phi))\,\underline{R}(M_\omega^{-1}(\phi))^{\mathrm{T}} \;, \end{aligned} \tag{8}$$

and in the direction of decreasing n as $F_{n-1}(\phi) = \underline{R}(\phi)^{-1} F_n(M_\omega(\phi))$, and $\underline{F}_{n-1}(\phi) = \underline{R}(\phi)^{-1} \underline{F}_n(M_\omega(\phi)) \underline{R}(\phi)$ respectively. Note that in the language of[7] $F_{n+1} = \mathcal{M}_\omega \underline{R} \cdot \mathcal{M}_\omega F_n$, where $\mathcal{M}_\omega$ is the Perron-Frobenius operator associated with M_ω.

The vector/tensor polarization/spin fields are **invariant**, if $F_{n+1} = F_n$, or $\underline{F}_{n+1} = \underline{F}_n$ respectively. In this case we take the opportunity to drop the index n. Thus we define the *invariant vector polarization field* (IvPF), the *invariant vector spin field* (IvSF), the *invariant tensor polarization field* (ItPF), and the *invariant tensor spin field* (ItSF) by

$$\text{IvPF:} \quad \vec{P}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\, \vec{P}(M_\omega^{-1}(\phi)) \tag{9}$$

$$\text{IvSF:} \quad \hat{N}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\, \hat{N}(M_\omega^{-1}(\phi)) \tag{10}$$

$$\text{ItPF:} \quad \underline{\bar{P}}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\, \underline{\bar{P}}(M_\omega^{-1}(\phi))\, \underline{R}(M_\omega^{-1}(\phi))^{\mathrm{T}} \tag{11}$$

$$\text{ItSF:} \quad \underline{\check{N}}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\, \underline{\check{N}}(M_\omega^{-1}(\phi))\, \underline{R}(M_\omega^{-1}(\phi))^{\mathrm{T}} \,. \tag{12}$$

Note that the trivial polarization fields $\vec{P}_{\mathrm{null}}(\phi) \equiv \vec{0}$ and $\underline{\bar{P}}_{\mathrm{null}}(\phi) \equiv \underline{0}$ are *always* invariant. They can, however *not* be normalized to generate spin fields. Also note that the IvPFs as well as the ItPFs form real linear spaces, i.g. if $\vec{P}_1$ and $\vec{P}_2$ are IvPFs, then so is $\mu\vec{P}_1 + \nu\vec{P}_2$ for real μ, ν.

Many more systems exist with “polarization-” or “spin-like” dynamics — and we will discover some in the course of the examples. Two natural questions arise:

(Q1) : Are these (and other) invariants somehow related for common M_ω, $\underline{R}$?
(Q2) : How do they change with changing M_ω, $\underline{R}$?

The new formalism mentioned in the title and described in the following sections helps at least in part answering these questions.

The interested reader might have discovered a certain clumsiness when it comes to describing vector/tensor polarization/spin fields efficiently at the same time. This is in part unavoidable when using the standard explicit formalism. When using our new formalism, the simultaneous treatment of these separate entities becomes almost trivial — since they all share the property that they are “polarization/spin-like”.

2. The New Formalism

In the Sec. 2.1 we will introduce the basic tools. We will do that in a slightly informal way in the sense that we will hide certain aspects of the framework. Although these aspects are in a way essential for a thorough understanding of the results, they are possibly not very useful for making first contact with the new formalism.

In Sec. 2.2 we will briefly discuss the hidden secret ingredient without going into too much detail since this would clearly be beyond the scope of this paper. The reader anxious to miss a detail is referred to the sources[1–6] in the references.

Finally Sec. 2.3 and 2.4 present two (maybe $2\frac{1}{2}$) theorems that I personally consider the *highlights* of our results in the sense that can particularly easily applied

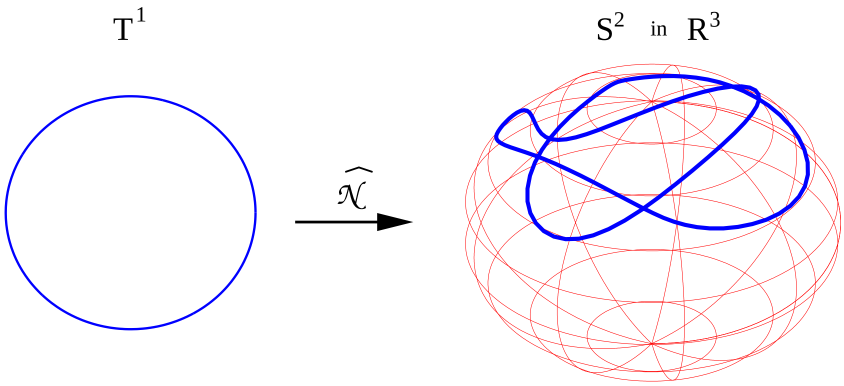

Fig. 1. Example of $\mathcal{C}^0$-IvSF driven by 1-d (vertical) orbital motion. The simulation was performed using the code `SPRINT` on a lattice of HERA–p.

to practical problems of spin dynamics and are in fact helpful for their discussion/solution. The sources[1–4] state further theorems which are more intricate and are therefore not treated in this overview paper.

2.1. *Basics*

$\mathbf{SO(3)}$-Action :

Let E be some "set" and

$$l \ : \ \mathbf{SO(3)} \times E \to E \ , \ (\underline{A}, x) \mapsto l(\underline{A}; x) \in E \ , \tag{13}$$

so that

$$\begin{aligned} l(\underline{1}; x) &= x \ \forall x \in E \\ l(\underline{A}_2\underline{A}_1; x) &= l(\underline{A}_2; l(\underline{A}_1; , x)) \ \forall x \in E \ \& \ \forall \underline{A}_1, \underline{A}_2 \in \mathbf{SO(3)} \ , \end{aligned} \tag{14}$$

then l is the $\mathbf{SO(3)}$-*action* of the $\mathbf{SO(3)}$-*Space* (E, l). If E is a linear space and l is linear in $\underline{A}$ & x, then l is called a *representation* of $\mathbf{SO(3)}$ on E. Actions can of course be defined any group: With any $\mathbf{SO(3)}$-map $\underline{A}$ and using the standard rules of matrix exponentiation, $g_{\underline{A}} : \mathbb{Z} \times E \to E$, $(n, x) \mapsto g_{\underline{A}}(n; x) := l(\underline{A}^n; x)$ is a $\mathbb{Z}$-action (for the group $(\mathbb{Z}, +)$). We can now immediately apply this concept to the spin-like systems that we know from Sec. 1

Vector pola.: $E_{\vec{\mathrm{v}}} := \mathbb{R}^3, \quad l_{\vec{\mathrm{v}}}(\underline{A}; \vec{s}) := \underline{A}\vec{s}$
Vector spin : $E_{\hat{\mathrm{v}}} := \mathcal{S}_2, \quad l_{\hat{\mathrm{v}}}(\underline{A}; \hat{s}) := \underline{A}\hat{s}$
Tensor pola.: $E_{\bar{\mathrm{t}}} := \{\underline{\bar{s}} \in \mathbb{R}^{3\times 3} : \underline{\bar{s}} = \underline{\bar{s}}^{\mathrm{T}}, \mathrm{trace}\underline{\bar{s}} = 0\}, \quad l_{\bar{\mathrm{t}}}(\underline{A}; \underline{\bar{s}}) := \underline{A}\,\underline{\bar{s}}\,\underline{A}^{\mathrm{T}}$
Tensor spin : $E_{\check{\mathrm{t}}} := \{\underline{\check{s}} \in E_{\bar{\mathrm{t}}} : \|\underline{\check{s}}\|_{\mathrm{t}} = 1\}, \quad l_{\hat{\mathrm{v}}}(\underline{A}; \underline{\check{s}}) := \underline{A}\,\underline{\check{s}}\,\underline{A}^{\mathrm{T}}$

If we substitute $\underline{R}$ for $\underline{A}$ we have formalized the transport mechanism for all 4 dynamics at once. Moreover we can write for the combined E/orbit-map K of

(E,l) with dynamics given by M_ω and $\underline{R}$. For E in $E_{\vec{\mathrm{v}}}$, $E_{\hat{\mathrm{v}}}$, $E_{\bar{\mathrm{t}}}$, $E_{\check{\mathrm{t}}}$, l in $l_{\vec{\mathrm{v}}}$, $l_{\hat{\mathrm{v}}}$, $l_{\bar{\mathrm{t}}}$, $l_{\check{\mathrm{t}}}$, and x in $\vec{s}$, $\hat{s}$, $\underline{\bar{s}}$, $\underline{\check{s}}$, :

$$K : \mathcal{T}^d \times E \to \mathcal{T}^d \times E, \ \begin{pmatrix} \phi \\ x \end{pmatrix} \mapsto \begin{pmatrix} M_\omega(\phi) \\ l(\underline{R}(\phi); x) \end{pmatrix} . \tag{15}$$

Now with $g_K : \mathbb{Z}\times\mathcal{T}^d\times E \to \mathcal{T}^d\times E$, $g_K(0;\phi,x) = (\phi,x)$, $(n,\phi,x) \mapsto g_K(n;\phi,x) := K(g_K(n-1;\phi,x)) = K^{-1}(g_K(n+1;\phi,x))$ is a recursively defined $\mathbb{Z}$-action which describes the time-discrete evolution of our dynamical system.

Furthermore for each field F out of $\vec{P}$, $\hat{N}$, $\underline{\bar{P}}$, $\underline{\check{N}}$, the invariance condition becomes:

$$F \circ M_\omega = l_{\mathrm{v}}(\underline{R}; F) \quad \text{ or equivalently} \tag{16}$$
$$F = l_{\mathrm{v}}(\underline{R} \circ M_\omega^{-1}; F \circ M_\omega^{-1}) = l_{\mathrm{v}}(\mathcal{M}_\omega \underline{R}; \mathcal{M}_\omega F) \ , \tag{17}$$

with $\mathcal{M}_\omega$ being the Perron-Frobenius operator corresponding to M_ω as in[7].

Let's add some more useful examples of $\mathbf{SO(3)}$-actions: First there is the *singlet* representation $E_{\mathrm{id}} = \mathbb{R}$, $l_{\mathrm{id}}(\underline{A}; \rho) := \rho$. It makes the evolution of a Liouville phase-space density (PSD) an $\mathbf{SO(3)}$-action:

$$\Psi_{n+1} = l_{\mathrm{id}}(\underline{R}; \Psi_n \circ M_\omega^{-1}) \equiv \Psi_n \circ M_\omega^{-1} \equiv \mathcal{M}_\omega \Psi_n \ , \tag{18}$$

i.p. for an *invariant* Liouville PSD the invariance condition reads $\Psi \circ M_\omega = l_{\mathrm{id}}(\underline{R}; \Psi) \equiv \Psi$.

Moreover, given two (E,l)-spaces ((E_1,l_1) and (E_2,l_2)) one may define the product space $E_{1\times 2} = E_1 \times E_2$,

$$l_{1\times 2}(\underline{A}; (x_1,x_2)) := (l_1(\underline{A}; x_1), l_2(\underline{A}; x_2)) \ . \tag{19}$$

$\underline{(E,l)\text{-Orbit } E_x \text{ of } x :}$

Now that we know how to operate an $\mathbf{SO(3)}$-map $\underline{A}$ on any $x \in E$, we will generalize the notion of an orbit: For all $x \in E$ we define the (E,l)-Orbit

$$E_x := l(\mathbf{SO(3)}; x) := \{l(\underline{A}; x) : \underline{A} \in \mathbf{SO(3)}\} \ . \tag{20}$$

We note that with g_K as define above, the (E, g_K)-orbit $g_K(\mathbb{Z};\phi,x) = \{K^{[n]}(\phi,x) : n \in \mathbb{Z}\}$, where $K^{[0]} = \mathrm{Id}$ and $K^{[n]} = K \circ K^{[n-1]} = K^{-1} \circ K^{[n+1]}$ is the more conventional notion of a (time-discrete) "orbit", namely the result of iterating an autonomous map infinitely often on some given starting point. Let us now consider the E-motion only and define $g_{K|E}(n,\phi;x)$ as the x component of $g_K(n;\phi,x)$. Then $g_{K|E}(\mathbb{Z},\phi;x) = \{l(\underline{R}^{[n]}(\phi);x) : n \in \mathbb{Z}\}$, with $\underline{R}^{[0]}(\phi) = \underline{1}$, $\underline{R}^{[n]}(\phi) = \underline{R}(\underline{M}_{n\omega}(\phi))\underline{R}^{[n-1]}(\phi) = \underline{R}(\underline{M}_{(n+1)\omega}(\phi))^{-1}\underline{R}^{[n+1]}(\phi)$. The $\underline{R}^{[n]}(\phi)$ generate either $\mathbf{SO(3)}$ or a subgroup thereof. Thus our new notion of an (E,l)-orbit is just a generalization of the conventional "orbit" of a dynamical system.

Some examples of (E,l)-orbits are:

$$l_{\vec{\mathrm{v}}}(\mathbf{SO(3)}; \vec{s}) \equiv \mathcal{S}_2 \cdot \|\vec{s}\|_2 \ , \ l_{\vec{\mathrm{v}}}(\mathbf{SO(3)}; \vec{0}) = \vec{0} \ , \ l_{\hat{\mathrm{v}}}(\mathbf{SO(3)}; \hat{s}) \equiv \mathcal{S}_2 \ . \tag{21}$$

It is clear that every E_x is a invariant set of $\mathbf{SO(3)}$, since $\forall \underline{A} \in \mathbf{SO(3)} : \underline{A}E_x = E_x$. Moreover, the $\mathcal{T}^d \times E_x$ are invariant sets of the combined "spin"-orbit[§] map K from Eqs. (4) and (15)

$$K(\mathcal{T}^d \times E_x) = \mathcal{T}^d \times E_x \ . \tag{22}$$

Isotropy Group :
We now come to a concept which is, in a way, conjugate to the (E,l)-orbits. For $x \in E$, the *subgroup* of $\mathbf{SO(3)}$ for which x is a *fixed point* of $l(\underline{A};\cdot)$ is called isotropy group of (E,l) at x:

$$\mathbf{Iso}(E,l;x) := \{\underline{A} \in \mathbf{SO(3)} : l(\underline{A};x) = x\} \ . \tag{23}$$

Generally, larger isotropy group at x means smaller orbit through x, i.p. $\mathbf{Iso}(E,l;x) = \mathbf{SO(3)}$ iff $E_x = \{x\}$. For example:

$$\mathbf{Iso}(E_{\check{\mathrm{v}}}, l_{\check{\mathrm{v}}}; \vec{0}) = \mathbf{SO(3)} \ , \ \mathbf{Iso}(E_{\check{\mathrm{v}}}, l_{\check{\mathrm{v}}}; \vec{s} \neq \vec{0}) = \{\text{rotations around } \vec{s}\} \cong \mathbf{SO(2)} \ . \tag{24}$$

We will need the isotropy groups for stating the Normal Form Theorem below.

G-Map (of $\mathbf{SO(3)}$) :
We now come to *structure preserving* maps between $\mathbf{SO(3)}$-spaces. Let $\Gamma : (E_1,l_1) \to (E_2,l_2)$, $x \mapsto \Gamma(x)$. If

$$l_2(\underline{A};\Gamma(x)) = \Gamma(l_1(\underline{A};x)) \ , \ \ \forall \underline{A} \in \mathbf{SO(3)} \ \& \ \forall x \in E_1 \ , \tag{25}$$

then Γ is called a **G**-map (of $\mathbf{SO(3)}$). We want to stress the point that a "**G**-map of $\mathbf{SO(3)}$" is not (in our notation) an "$\mathbf{SO(3)}$-map" which is just a map $A \in \mathbf{SO(3)}$, while Γ in general $\notin \mathbf{SO(3)}$! As an example we take $E_{\hat{\mathrm{v}}}$ and $E_{\check{\mathrm{t}}}$. Then $\Gamma_{\check{\mathrm{t}} \leftarrow \hat{\mathrm{v}}} : E_{\hat{\mathrm{v}}} \to E_{\check{\mathrm{t}}}$,

$$\hat{s} \mapsto \Gamma_{\check{\mathrm{t}} \leftarrow \hat{\mathrm{v}}}(\hat{s}) := \sqrt{\frac{3}{2}} \left(\hat{s}\hat{s}^{\mathrm{T}} - \frac{1}{3} \underline{1} \right) \tag{26}$$

is indeed a **G**-map from $E_{\hat{\mathrm{v}}}$ to $E_{\check{\mathrm{t}}}$. The concept of **G**-maps will become essential in stating the Decomposition Theorems below. The **G**-map $\Gamma_{\check{\mathrm{t}} \leftarrow \hat{\mathrm{v}}}$ will be used in example 1.

2.2. ***The Secret Ingredient: Global Regularity Constraints***

The framework so far was only described technically. In order to state and prove theorems regularity constraints are needed. The kind of regularity chosen here is **global continuity**. The "sets" involved are in fact topological spaces, i.e. sets that have been explicitly assigned a topology. Moreover the involved functions/maps/fields/... need to be continuous *everywhere*[¶] on E. For example:

[§] We are aware that we are overusing "orbit" here for the orbital phases on one side and for the result of a group acting on a set on the other side. We think, however that the distinction is quite clear from the context.

[¶] not just in some environment of certain points of interest!

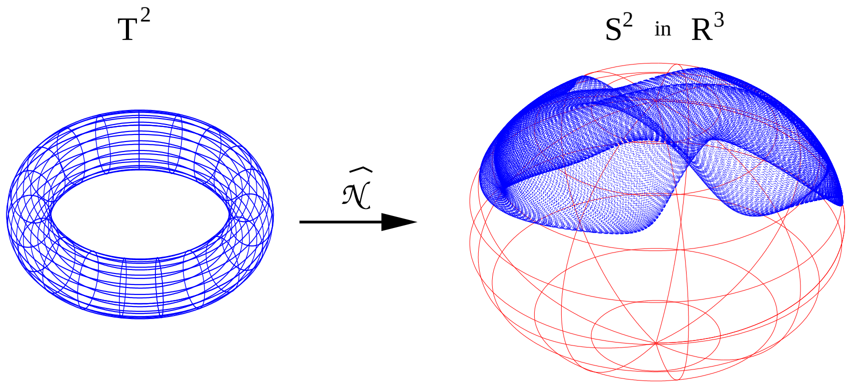

Fig. 2. Example of $\mathcal{C}^0$-IvSF driven by 2-d (vertical & horizontal) orbital motion. The simulation was performed using the code `SPRINT` on a lattice of HERA–p.

$\underline{R} \in \mathcal{C}^0(\mathcal{T}^d, \mathbf{SO(3)})$, $M_\omega \in \mathrm{Homeo}(\mathcal{T}^d)$, i.e. $M_\omega \in \mathcal{C}^0(\mathcal{T}^d, \mathcal{T}^d)$, $M_\omega \in \mathrm{aut}(\mathcal{T}^d)$ (is globally invertible), and $M_\omega^{-1} \in \mathcal{C}^0(\mathcal{T}^d, \mathcal{T}^d)$. Furthermore all our (invariant) fields (and candidates) must be *globally* continuous. To indicate this we will from now on call them:

$\mathcal{C}^0$-IvPF : $(\vec{P} \in \mathcal{C}^0(\mathcal{T}^d, E_{\vec{\mathrm{v}}}))$,
$\mathcal{C}^0$-IvSF : $(\hat{N} \in \mathcal{C}^0(\mathcal{T}^d, E_{\hat{\mathrm{v}}}))$,
$\mathcal{C}^0$-ItPF : $(\underline{\bar{P}} \in \mathcal{C}^0(\mathcal{T}^d, E_{\bar{\mathrm{t}}}))$, and
$\mathcal{C}^0$-ItSF : $(\underline{\check{N}} \in \mathcal{C}^0(\mathcal{T}^d, E_{\check{\mathrm{t}}}))$, or just
invariant (E, l)-field : $(F \in \mathcal{C}^0(\mathcal{T}^d, E))$.

We note here that *global* continuity is, in a way, a *strong* restriction (see Sec. 3.4), but we also note that global *continuity* is, in a way, a *weak* restriction, since with very few exceptions, functions realized in physics normally tend to be *smooth* ($\mathcal{C}^\infty$) except on a countable set of lower dimensional "cracks". It may well be that a system has a, say $\mathcal{C}^0$-IvSF, but that the invariant is nowhere $\mathcal{C}^1$. — Then the $\mathcal{C}^0$-IvSF would possibly fail to generate nice adiabatic invariants (as needed for Froissart–Stora[8,9]) and it could be numerically ill-conditioned which means that numerically no or no stable approximation to an IvSF (assumed piece-wise smooth but not globally continuous) could be found. Our framework does not make (in the way it is stated) any statements about fields which are not $\mathcal{C}^0$. However, the regularity constraints could be made stronger ($\to$ globally $\mathcal{C}^k$, $k > 0$) or weaker ($\to$ only measurable), thereby potentially modifying the applicability of the premises and the strength of the conclusions of our theorems.

2.3. *The Normal Form Theorem (NFT)*

The following theorem nicely states a criterion that relates invariant fields with normal forms — for all possible combinations of E and l. Let $\underline{T} \in \mathcal{C}^0(\mathcal{T}^d, \mathbf{SO(3)})$, (E, l),

M_ω, $\underline{R}$ as before, and $x \in E$ fixed. Define $F \in \mathcal{C}^0(\mathcal{T}^d, E)$, and $\underline{R}' \in \mathcal{C}^0(\mathcal{T}^d, \mathbf{SO(3)})$ by

$$F := l(\underline{T}; x) \ , \ \underline{R}' := \underline{T}^{\mathrm{T}} \circ M_\omega \ \underline{R} \ \underline{T} \ . \tag{27}$$

Then F is an invariant (E, l)-field ($F \circ M_\omega = l(\underline{R}; F)$), iff

$$\underline{R}'(\phi) \in \mathbf{Iso}(E, l; x) \ \forall \phi \in \mathcal{T}^d \ . \tag{28}$$

The proof can be found in [1,2,4]. If $\underline{R}'$ is element of some proper subgroup of $\mathbf{SO(3)}$, then $(M_\omega, \underline{R}')$ is called a normal form of $(M_\omega, \underline{R})$. The NFT answers (Q2) from Sec. 1.2 to the extent that invariant fields can be related to transformed dynamics $\underline{R}'$ that look simpler than (are normal forms of) the original dynamics $\underline{R}$.

Now take for example vector spin motion on $(E_{\hat{\mathrm{v}}}, l_{\hat{\mathrm{v}}})$ with $\hat{x} := (0, 0, 1)^{\mathrm{T}}$. Then $\hat{N} := l_{\hat{\mathrm{v}}}(\underline{T}; \hat{x})$ is a $\mathcal{C}^0$-IvSF for some $\underline{T}$ iff $\underline{R}'(\phi) = \underline{T}(M_\omega(\phi))^{\mathrm{T}} \ \underline{R}(\phi) \ \underline{T}(\phi) \in \mathbf{SO(2)} \ \forall \phi$. As a matter of fact it is potentially simpler to proof that the transformed dynamics $\underline{R}'$ lies in $\mathbf{SO(2)} \subset \mathbf{SO(3)}$ than showing that $(\hat{N} \circ M_\omega)(\phi) = l(\underline{R}(\phi); \hat{N}(\phi))$, $\forall \phi$. However, one has to admit that no significant progress towards general existence theorems for $\mathcal{C}^0$-IvSFs has been made since the NFT was stated first in [1,2].

2.4. *Decomposition Theorems*

The following lemma and corollary allow to relate various realizations of the same dynamics. In my opinion they are of greatest practical interest since the enable the (rigorous) generation of a multitude of useful invariants from, say a $\mathcal{C}^0$-IvSF, at only little extra cost. In particular the SML/DC completely and consistenly answer (Q1) from Sec. 1.2.

The $\mathbf{SO(3)}$ Mapping Lemma (SML): Let $\Gamma \in \mathcal{C}^0((E_1, l_1), (E_2, l_2))$ be a $\mathbf{G}$-map (of $\mathbf{SO(3)}$) from (E_1, l_1) to (E_2, l_2), $f_1 \in \mathcal{C}^0(\mathcal{T}^d, E_1)$, and $f_2 \in \mathcal{C}^0(\mathcal{T}^d, E_2)$ be defined by $f_2 := \Gamma \circ f_1$. Then

$$l_2(\underline{R}; f_2) = \Gamma(l_1(\underline{R}; f_1)) \tag{29}$$

for all M_ω, $\underline{R}$, i.e. the field dynamics from (E_1, l_1) is induced on (E_2, l_2) by Γ. In particular $f_1 \circ M_\omega = l_1(\underline{R}, f_1) \Rightarrow f_2 \circ M_\omega = l_2(\underline{R}, f_2)$, in other words, given that f_1 is an invariant (E_1, l_1)-field, then $f_2 = \Gamma \circ f_1$ is an invariant (E_2, l_2)-field.

The proof [1,2,4] is too short to be omitted: The field f_1 evolves on (E_1, l_1) via $f_1 \mapsto f_1' = l_1(\underline{R} \circ M_\omega^{-1}; f_1 \circ M_\omega^{-1})$ the field f_2 evolves on (E_2, l_2) via $f_2 \mapsto f_2' = l_2(\underline{R} \circ M_\omega^{-1}; f_2 \circ M_\omega^{-1})$. By definition of f_2 we have $f_2 = \Gamma \circ f_1$. Then by definition of the $\mathbf{G}$-map γ it follows that $f_2' = l_2(\underline{R} \circ M_\omega^{-1}; \Gamma \circ f_1 \circ M_\omega^{-1}) = \Gamma(l_1(\underline{R} \circ M_\omega^{-1}; f_1 \circ M_\omega^{-1})) = \Gamma \circ f_1'$, The claim follows from M_ω being invertible; q.e.d.

If $\Gamma \in \mathrm{Homeo}((E_1, l_1), (E_2, l_2))$, then also the converse is true, and thus f_1 is an invariant (E_1, l_1)-field iff f_2 is an invariant (E_2, l_2)-field.

We only mention here that the Decomposition Corollary (DC) generalizes the SML to $\mathbf{G}$-maps from (E_1, l_1)-orbits E_{1,x_1} with arbitrary $x_1 \in E_1$ to (E_2, l_2)-orbits E_{2,x_2} with arbitrary $x_2 \in E_2$.

3. Examples

3.1. *Example 1: Relation $\mathcal{C}^0$-IvSF $\leftrightarrow$ $\mathcal{C}^0$-ItSF*

The **G**-map $\Gamma_{\check{t}\leftarrow\hat{v}}$ from Eq. (26) is in $\mathcal{C}^0(E_{\hat{v}}, E_{\check{t}})$. Thus the SML applies and hence $\underline{\check{N}} := \Gamma_{\check{t}\leftarrow\hat{v}}(\hat{N})$ is a $\mathcal{C}^0$-ItSF, if $\hat{N}$ is a $\mathcal{C}^0$-IvSF.

In the case that the constructed $\mathcal{C}^0$-ItSF has 2 distinct eigenvalues, then if the $\mathcal{C}^0$-IvSF is unique up to global sign, so is the $\mathcal{C}^0$-ItSF[6].

In order to construct tensor spin fields that have 3 distinct eigenvalues we define $E_\perp := \{(\hat{u}, \hat{v}) \in E_{\hat{v}} \times E_{\hat{v}} : \hat{u} \perp \hat{v}\}$ and then $\Gamma^{(\alpha,\beta)}_{3\text{ev}} : E_\perp \to E_{\check{t}}$

$$(\hat{f}, \hat{g}) \mapsto \alpha\underline{1} - (2\alpha + \beta)\hat{f}\hat{f}^{\mathrm{T}} + (\beta - \alpha)\hat{g}\hat{g}^{\mathrm{T}} \tag{30}$$

with $\alpha \in (0, \infty)$, $-\alpha/2 < \beta < \alpha$, $\alpha^2 + \alpha\beta + \beta^2 = 1/2$[‖] is a **G**-map in $\mathcal{C}^0(E_\perp, E_{\check{t}})$. Now assume $\hat{N}_1$ is a $\mathcal{C}^0$-IvSF and thus $\Gamma_{\check{t}\leftarrow\hat{v}}(\hat{N}_1)$ is a $\mathcal{C}^0$-ItSF. It is clear that an invariant $(E_\perp, l_{\hat{v}\times\hat{v}})$-field needs a second independent $\mathcal{C}^0$-IvSF $\hat{N}_2 \perp \hat{N}_1$. We have shown in[6] that $\Gamma^{(\alpha,\beta)}_{3\text{ev}}(\hat{N}_1, \hat{N}_2)$, $\hat{N}_1 \perp \hat{N}_2$ can only be a $\mathcal{C}^0$-ItSF, if the the vector spin system is on *spin-orbit resonance*, i.e. when the $\mathcal{C}^0$-IvSF is non-unique (by more than just a global sign). The condition $\hat{N}_1 \perp \hat{N}_2$ is no restriction because if $\hat{N}_1$ and $\hat{N}_2$ are linearly independent then two non-zero $\mathcal{C}^0$-IvPFs $\vec{P}_\parallel$ and $\vec{P}_\perp$ exist with $\hat{N}_2 = \vec{P}_\parallel + \vec{P}_\perp$ and $\vec{P}_\perp$ can be normalized to give $\hat{N}_\perp \perp \hat{N}_1$.

If a $\mathcal{C}^0$-ItPF has only 1 eigenvalue, it must be zero and thus the $\mathcal{C}^0$-ItPF can only be the trivial one $(\underline{0})$.

3.2. *Example 2: Spin-1/2 Density Matrix*

The physics-interface between the macroscopic, classical description of a particle beam in an accelerator, and a quantum (scattering) processes is the density matrix $\underline{\rho}$. (We will use $\underline{\rho}^{(1/2)}$ for spin-1/2 particles, and $\underline{\rho}^{(1)}$ for spin-1 particles.) In a collider the beam might change from turn to turn and so $\underline{\rho} = \underline{\rho}_n$. The expectation value of an arbitrary observable represented by the operator $\mathcal{O}$ at turn n is given by $\langle\mathcal{O}\rangle(n) = \int d\phi\, dJ\ \text{trace}\,(\underline{\rho}_n\mathcal{O})$. Our framework handles arbitrary fixed sets of orbital actions $J = \text{const}$. The description of the beam then follows from integrating over all actions. Hence,

$$\underline{\rho}^{(1/2)}_J(\phi) := \Psi_J(\phi)\frac{1}{2}\Big(\underline{1} + \underline{\vec{\sigma}} \cdot \vec{S}_J(\phi)\Big)\ , \tag{31}$$

where Ψ_J is the (orbital) Liouville PSD and $\vec{S}_J$ the polarization field describing the beam, and $\underline{\vec{\sigma}}$ is the vector of Pauli matrices. In the following we will suppress any reference to J. $\underline{\rho}^{(1/2)}$ is a complex self-adjoint matrix $\underline{\rho}^{(1/2)} \in E_{1/2} := \{\underline{r} \in \mathbb{C}^{2\times 2} : \underline{r}^\dagger = \underline{r}\}$. We define $\Gamma_{1/2} : E_{\text{id}} \times E_{\vec{v}} \to E_{1/2}$,

$$(\psi, \vec{s}) \mapsto \Gamma_{1/2}(\psi, \vec{s}) := \frac{1}{2}\big(\psi\underline{1} + \underline{\vec{\sigma}} \cdot \vec{s}\big)\ , \tag{32}$$

[‖] A possible solution is $\alpha = 1/2$, $\beta = (\sqrt{5} - 1)/4$.

and $l_{1/2}$ via Eq. (32) and

$$l_{1/2}(\underline{A};\Gamma_{1/2}(\psi,\vec{s})) := \Gamma_{1/2}(l_{\rm id}(\underline{A};\psi), l_{\vec{v}}(\underline{A};\vec{s})) \ . \tag{33}$$

Then, with

$$\underline{\rho}^{(1/2)} = \Gamma_{1/2}(\Psi, \Psi\vec{S}) \tag{34}$$

$\Gamma_{1/2} \in \mathrm{Homeo}(E_{\rm id}\times E_{\vec{v}}, E_{1/2})$ is $\mathbf{G}$-map.

This implies that $\underline{\rho}^{(1/2)}_{\rm equi} = \Gamma_{1/2}(\Psi_{\rm equi}, \Psi_{\rm equi}\vec{P})$ is an invariant $(E_{1/2}, l_{1/2})$-field, iff $\Psi_{\rm equi}$ is an invariant Liouville PSD and $\vec{P}$ is a $\mathcal{C}^0$-IvPF. Only in such an equilibrium state, $\underline{\rho}$ and thus $\langle\mathcal{O}\rangle$ become independent of n.

The maximum attainable equilibrium polarization state is realized when $\underline{\rho}^{(1/2)}_{\rm equi} \to \Gamma_{1/2}(\Psi_{\rm equi}, \Psi_{\rm equi}\hat{N})$, i.e. when $\hat{N}$ is a $\mathcal{C}^0$-IvSF.

3.3. *Example 3: Spin-1 Density Matrix*

The density matrix for a beam of spin-1 particles is slightly more intricate since spin-1 particles in addition to carrying a vector polarization also carry a tensor polarization.

$$\underline{\rho}^{(1)} := \Psi\frac{1}{3}\Big(\underline{1} + \frac{3}{2}\vec{\underline{\Sigma}}\cdot\vec{S} + \sqrt{\frac{3}{2}}\sum_{i,j=1}^{3}\bar{\underline{S}}_{ij}(\underline{\Sigma}_i\,\underline{\Sigma}_j + \underline{\Sigma}_j\,\underline{\Sigma}_i)\Big) \ , \tag{35}$$

with

$$\underline{\Sigma}_{1,2,3} := \sqrt{\frac{1}{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{36}$$

Also $\underline{\rho}^{(1)}$ is a complex self-adjoint matrix $\underline{\rho}^{(1)} \in E_1 := \{\underline{r} \in \mathbb{C}^{3\times 3} : \underline{r}^\dagger = \underline{r}\}$. Now we define $\Gamma_1 : E_{\rm id}\times E_{\vec{v}} \times E_{\bar{t}} \to E_1$,

$$\Gamma_1(\psi,\vec{s},\underline{\bar{s}}) := \frac{1}{3}\Big(\psi\underline{1} + \frac{3}{2}\vec{\underline{\Sigma}}\cdot\vec{s} + \sqrt{\frac{3}{2}}\sum_{i,j=1}^{3}\bar{\underline{s}}_{ij}(\underline{\Sigma}_i\,\underline{\Sigma}_j + \underline{\Sigma}_j\,\underline{\Sigma}_i)\Big) \ , \tag{37}$$

$$l_1(\underline{A};\Gamma_1(\psi,\vec{s},\underline{\bar{s}})) := \Gamma_1(l_{\rm id}(\underline{A};\psi), l_{\vec{v}}(\underline{A};\vec{s}), l_{\bar{t}}(\underline{A};\underline{\bar{s}})) \ , \tag{38}$$

and

$$\underline{\rho}^{(1)} = \Gamma_1(\Psi, \Psi\vec{S}, \Psi\underline{\bar{S}}) \ . \tag{39}$$

Then $\Gamma_1 \in \mathrm{Homeo}(E_{\rm id}\times E_{\vec{v}}\times E_{\bar{t}}, E_1)$ is $\mathbf{G}$-map, and hence $\underline{\rho}^{(1)}_{\rm equi} = \Gamma_1(\Psi_{\rm equi}, \Psi_{\rm equi}\vec{P}, \Psi_{\rm equi}\underline{\bar{P}})$ is an invariant (E_1, l_1)-field, iff $\Psi_{\rm equi}$ is an invariant Liouville PSD and $\vec{P}$ is a $\mathcal{C}^0$-IvPF and $\underline{\bar{P}}$ is a $\mathcal{C}^0$-ItPF.

The maximum attainable equilibrium polarization state is realized, when $\underline{\rho}^{(1)}_{\rm equi} \to \Gamma_1(\Psi_{\rm equi}, \Psi_{\rm equi}\hat{N}, \Psi_{\rm equi}\underline{\check{N}})$, i.e. when $\hat{N}$ is a $\mathcal{C}^0$-IvSF and and $\underline{\check{N}}$ is a $\mathcal{C}^0$-ItSF.

3.4. *A Discontinuous Example (4)*

The model that we discuss here is slightly artificial in the sense that the orbital motion has to be exactly *on-resonance* — for a moderately low-order resonance. We study only vertical motion $d = 1$ in a storage ring with 2 Siberian Snakes. M_ω is chosen resonant, i.e. $\omega_y \equiv Q_y^{(n)} = \frac{1}{4n-2}$, $n = 1, 2, 3, \ldots$ The model of the ring is the so called *single resonance model*[10–12] equipped with a Lee–Courant 2-snake scheme[13–16]. It consists of 2 Siberian snakes 180° in azimuth apart, where both snake axes are in the mid-plane and the axes are perpendicular. Lee–Courant schemes are among those that keep the unperturbed $\hat{n}_0$-axis, i.e. the invariant vector spin field on the design orbit (for J=0) strictly vertical and fix the design orbit spin tune to $\nu_0 = 1/2$. However, for $n \in \mathbb{N}$ orbital tune and the design orbit spin tune fulfill a $(2n-1)$st order resonance condition $\nu_0 = (2n+1)Q_y^{(n)}$. As it turns out[2,17], in order to be single valued under the evolution by the one turn map, the IvSF needs $2n$ jump-discontinuities (*local* sign flips) at phases distributed uniformly on the torus. Therefore the IvSF is not a $\mathcal{C}^0$-IvSF, and thus the framework presented here does not apply. However, the corresponding ItSF **is** a $\mathcal{C}^0$-ItSFand thus for the $\mathcal{C}^0$-ItSF the framework **does** apply. Note that $\Gamma_{\check{\mathrm{t}}\leftarrow\hat{\mathrm{v}}}$ from example 1 (Sec. 3.1) is not bijective (it's more or less a quadratic form), so that the existence of a $\mathcal{C}^0$-ItSF does not imply the existence of a $\mathcal{C}^0$-IvSF.

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